

COMBINATORIAL PROPERTIES OF POLY-BERNOULLI RELATIVES

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ABSTRACT. In this note we augment the poly-Bernoulli family with two new combinatorial objects. We derive formulas for the relatives of the poly-Bernoulli numbers using the appropriate variations of combinatorial interpretations. Our goal is to show connections between the different areas where poly-Bernoulli numbers and their relatives appear and give examples how the combinatorial methods can be used for deriving formulas between integer arrays.

1. INTRODUCTION

Poly-Bernoulli numbers were introduced by M. Kaneko [24] in 1997 as a generalization of the classical Bernoulli numbers during his investigations of multiple zeta values. The sequence received attention because of its nice properties, that were proved by several authors analytically. The importance of the notion of the poly-Bernoulli numbers is underlined also by the fact that there are several drastically different combinatorial interpretations [6]. The combinatorics of the family of poly-Bernoulli numbers is shown in the bijections that can be described between the sets. These bijections help us to understand more the properties of the poly-Bernoulli numbers.

In this paper we consider two number arrays that are relatives of poly-Bernoulli numbers. The importance of the attention in this direction is that in some combinatorial problems these relatives arise naturally. Also Kaneko's number theoretical investigations led to these numbers. We go through the known combinatorial interpretations of poly-Bernoulli numbers [6] and for most of them we show that slight modifications of the original combinatorial definition lead to the descriptions of the two related sequences.

This way we connect poly-Bernoulli numbers to the class of permutations with a special excedance set. We augment the list of poly-Bernoulli families with two classes of 01 matrices defined by a given forbidden set of submatrices.

The outline of the paper is as follows. After a short introduction of the poly-Bernoulli numbers we define the poly-Bernoulli relatives using the well known interpretation of lonesum matrices. We derive different formulas for these arrays and show relations between the number sequences using appropriate combinatorial interpretations. We close our discussion with a conjecture related to the central binomial sum.

1.1. Poly-Bernoulli numbers. The story of the Bernoulli numbers starts with investigating the sum of the m th powers of the first n positive integer that are polynomials in n . Jacob Bernoulli recognized the scheme in the coefficients of these polynomials. Kaneko generalized the well known generating function of the Bernoulli numbers and defined the poly-Bernoulli numbers.

Definition 1. [25] *Poly-Bernoulli numbers (denoted by $B_n^{(k)}$, where n is a positive integer and k is an integer) are defined by the following exponential generating function*

$$(1) \quad \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{1 - e^{-x}},$$

where

$$Li_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k},$$

i.e. $Li_k(z)$ is the k th poly-logarithm when $k > 0$ and a rational function when $k \leq 0$.

From the combinatorial point of view we are interested only in the poly-Bernoulli numbers with negative k indices since in this case the numbers form an array of positive integers. From now on we mean poly-Bernoulli numbers always with negative indices even if we don't emphasize it explicitly. For the sake of convenience we denote in the rest of the paper $B_n^{(-k)}$ as $B_{n,k}$. The following table shows the values of poly-Bernoulli numbers for small indices. An extended array can be find in OEIS [33] A099594.

TABLE 1. The poly-Bernoulli numbers $B_{n,k}$

$\begin{smallmatrix} k \\ n \end{smallmatrix}$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

The symmetry of the array in n and k is immediately conspicuous. Analytically this property is obvious from the symmetry of the double exponential function:

$$(2) \quad \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

Three formulas of poly-Bernoulli numbers were proved combinatorially in the literature:

the combinatorial formula ([8], [6])

$$(3) \quad B_{n,k} = \sum_{m=0}^{\min(n,k)} m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} m! \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\},$$

an inclusion-exclusion type formula ([8])

$$(4) \quad B_{n,k} = (-1)^n \sum_{m=0}^n (-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (m+1)^k,$$

and a recursion ([6])

$$(5) \quad B_{n,k+1} = B_{n,k} + \sum_{m=1}^n \binom{n}{m} B_{n-(m-1),k}.$$

One of the first (and widely known) combinatorial interpretation of the poly-Bernoulli numbers are lonesum matrices [8]. Lonesum matrices arise in the roots of discrete tomography. Ryser [35] investigated in the late 1950's the problem of the reconstruction of a matrix from given row and column sums. The 01 matrices that are uniquely reconstructible from their row and column sum vectors are called *lonesum* matrices. We denote the set of lonesum matrices of size $n \times k$ as \mathcal{L}_n^k . Note that we allow $n = 0$ (and $k = 0$ too), in which case the empty matrix counted as lonesum.

Theorem 1. [8] *The number of 01 lonesum matrices of size $n \times k$ is given by the poly-Bernoulli numbers of negative k indices.*

$$|\mathcal{L}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} = B_{n,k}.$$

Proof. (Sketch) Take a lonesum matrix M of size $n \times k$. Add a new column and new row with all 0 entries and obtain lonesum matrix \widehat{M} of size $(n+1) \times (k+1)$. We know that \widehat{M} contains at least one all-0 row and at least one all-0 column (this information was not known for M). Partition the rows and the columns according to the sum of its entries. In the case of lonesum matrices ‘having the same row/column sum’ and ‘being equal’ is the same relation. Easy to see that the number of row classes will be the same as the number of equivalence classes of columns. We denote this common value by $m+1$. The plus 1 stands for the class of extra row/column, the class of all-0 rows and all-0 columns. The row sums order the (m) many classes of not all-0 rows. Similarly the column sums order the classes of not all-0 columns. Our formula comes from the fact that from the two partitions and two orders it is easy to decode M . \square

This important theorem started the combinatorial investigations of poly-Bernoulli numbers. From the point of combinatorics $B_{n,k} = |\mathcal{L}_n^k|$ is the natural way of defining the poly-Bernoulli numbers. It turned out that there are several alternative combinatorial ways to describe the poly-Bernoulli numbers. Some of them were investigated before Kaneko's pioneering work. Next we define two related 2-dimensional sequences combinatorially.

1.2. PB-Relatives. We consider lonesum matrices with further restrictions on the occurrence of all-0 columns resp. all-0 rows. More precisely let $\mathcal{L}_n^k(c|)$ denote the set of lonesum matrices with the property that each column contains at least one 1 entry and $\mathcal{L}_n^k(c|r|)$ the set of lonesum matrices with the property that each column and each row contains at least one 1.

Definition 2. $C_{n,k}$ denotes $|\mathcal{L}_n^k(c)|$, i.e. the number of lonesum matrices of $n \times k$ without all-0 columns.

$D_{n,k}$ denotes $|\mathcal{L}_n^k(c|r)|$, i.e. the number of lonesum matrices of $n \times k$ without all-0 columns and all-0 rows.

Let us see the first few values of our new numbers:

TABLE 2. Poly-Bernoulli relatives: $C_{n,k}$ and $D_{n,k}$

$\begin{smallmatrix} k \\ n \end{smallmatrix}$	0	1	2	3	4
1	1	1	1	1	1
2	1	3	7	15	31
3	1	7	31	115	391
4	1	15	115	675	3451
5	1	31	391	3451	25231

$\begin{smallmatrix} k \\ n \end{smallmatrix}$	1	2	3	4	5
1	1	1	1	1	1
2	1	5	13	29	61
3	1	13	73	301	1081
4	1	29	301	2069	11581
5	1	61	1081	11581	95401

First we give combinatorial formulas for the two new numbers:

Theorem 2. *We have*

(i) for $n \geq 1$ and $k \geq 0$

$$C_{n,k} = |\mathcal{L}_n^k(c)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \left\{ \begin{matrix} k \\ m \end{matrix} \right\},$$

(ii) for $n \geq 1$ and $k \geq 1$

$$D_{n,k} = |\mathcal{L}_n^k(c|r)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} k \\ m \end{matrix} \right\}.$$

Proof. (Sketch) (i): We have the information that there is no all-0 column. Hence we do not need the extra column. The extra row ensures that the extended matrix has the class of all-0 rows. m denotes the number of classes of (k many non-0) columns. $m+1$ will be the number of classes of the $n+1$ rows. The rest is a straightforward repeat of the original argument.

(ii) is immediate by the same logic. \square

From the combinatorial definition it is obvious that the series $B_{n,k}$ and $D_{n,k}$ are symmetric in n and k . The symmetry of the $C_{n,k}$ numbers ($C_{n,k} = C_{k+1,n-1}$) is also transparent from our table. But its proof is not straightforward. We present it in a latter section.

The combinatorial definitions make it clear that the sequence $C_{n,k}$ is the binomial transform of $D_{n,k}$ and the sequence $B_{n,k}$ is the binomial transform of $C_{n,k}$. Precisely:

Observation 1. *The following relations hold*

(i)

$$B_{n,k} = 1 + \sum_{i=1}^k \binom{k}{i} C_{n,i} = \sum_{i=0}^k \binom{k}{i} C_{n,i}, \quad (k \geq 0, n \geq 1),$$

(ii)

$$C_{n,k} = \sum_{i=1}^n \binom{n}{i} D_{i,k}, \quad (k \geq 1, n \geq 1),$$

(iii)

$$B_{n,k} = 1 + \sum_{i=1}^n \sum_{j=1}^k \binom{n}{i} \binom{k}{j} D_{i,j}, \quad (k \geq 1, n \geq 1).$$

Proof. (i): To describe an arbitrary non-0 lonesum matrix we need to identify all its columns with at least one 1 (their number is denoted by $i(> 0)$) and their entries in these i columns (that is describing a lonesum matrix of size $n \times i$ that contains at least one 1 in each column). This simple fact proves the formula of (i).

We obtain (ii) following the same argument on rows. (iii) summarizes (i) and (ii). \square

There are other connections, recursions, combinatorial properties of the poly-Bernoulli numbers and its two relatives. They are not obvious, we discuss them when the appropriate combinatorial interpretations appear below.

These numbers seem to be just minor modifications of the original poly-Bernoulli numbers. In spite of the first impression, it turns out that these numbers appeared a natural way in earlier papers. Now we summarize the analytical properties of these numbers (obtained by others). The rest of the paper is combinatorial.

1.3. Analytical results in the literature. Arakawa and Kaneko [4] introduced a function that are referred in the literature as the Arakawa-Kaneko function.

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt.$$

The values of this function at non-positive integers are given by

$$\xi_k(-m) = (-1)^m C_m^{(k)},$$

where the generating function of the numbers $\{C_n^{(k)}\}$ (for arbitrary integers k) is given by

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{e^x - 1}.$$

They computed the double exponential generating function of the $C_n^{(-k)}$ numbers. We know that the exponential functions of two number sequences differ only by an e^x (resp. e^y) factor when one sequence is the binomial transform of the other. From this observation we can conclude the binomial transformation relation between poly-Bernoulli numbers and $\{C_n^{(-k)}\}$. It is immediate that $C_{n,k} = C_n^{(-k)}$. Furthermore we can obtain the generating function of $D_{n,k}$ numbers.

Theorem 3. (i)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^x}{e^x + e^y - e^{x+y}},$$

(ii)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} D_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{1}{e^x + e^y - e^{x+y}}.$$

Kaneko realized the importance of the C-relative and in a recent paper [26] summarized formulas and properties of $B_n^{(k)}$ and $C_n^{(k)}$ parallel. Kaneko showed also a simple arithmetic connection between the two series.

$$B_{n,k} = C_{n,k} + C_{n+1,k-1}$$

In our investigations we show this relation combinatorially using the variations of the so called Callan permutations. Moreover we prove a similar relation between the series $D_{n,k}$ and $C_{n,k}$.

2. 01 MATRICES WITH EXCLUDED SUBMATRICES

The study of matrices that are characterized by excluded submatrices is an active research area with many important results and applications [31]. Given two matrices A and B we say that A avoids B whenever A does not contain B as a submatrix. (Given a matrix M a submatrix is a matrix that can be obtained from M by deletion of rows and columns.)

Generally we can set the following problem: Let $S = \{M_1, \dots, M_r\}$ be a set of 01 matrices. $\mathcal{M}_n^k(S)$ denote the $n \times k$ 01 matrices that do not contain any matrix of the set S , $\mathcal{M}_n^k(S; c|)$ denote these matrices with the extra condition of containing in any column at least one 1, and $\mathcal{M}_n^k(S; r|c|)$ denote those with the same extra condition on rows also.

Lonesum matrices can be characterized also with the terminology of forbidden submatrices [35]. Lonesum matrices are matrices that avoid the following set of submatrices:

$$L = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

I.e. $\mathcal{L}_n^k = \mathcal{M}_n^k(L)$, $\mathcal{L}_n^k(c|) = \mathcal{M}_n^k(L; c|)$ and $\mathcal{L}_n^k(r|c|) = \mathcal{M}_n^k(L; r|c|)$.

Interestingly beyond the set L there are other matrix sets S which forbiddance as submatrices lead to the poly-Bernoulli numbers.

2.1. Γ -free matrices and recursions. In [6] the authors investigated the so called Γ -free matrices, matrices with the forbidden set:

$$\Gamma = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

and showed bijectively that the number of $n \times k$ Γ -free matrices (their set is denoted by \mathcal{G}_n^k) are the $B_{n,k}$ poly-Bernoulli numbers. Clearly the forbiddance of all 0 rows/resp. columns has the same effect in this case as in the case of the lonesum matrices.

Theorem 4. *We have*

(i)

$$|\mathcal{G}_n^k(c)| = C_{n,k},$$

(ii)

$$|\mathcal{G}_n^k(r|c)| = D_{n,k}.$$

The structure of these matrices gives a transparent explanation of the recursive formula of poly-Bernoulli numbers that was first proven by Kaneko [4]. In the same spirit we can establish the recursions concerning the poly-Bernoulli relatives.

Theorem 5. (i)

$$C_{n,k+1} = \sum_{m=1}^n \binom{n}{m} C_{n-m+1,k},$$

(ii)

$$D_{n,k+1} = \sum_{m=1}^n \binom{n}{m} (D_{n-m,k} + D_{n-m+1,k}).$$

Proof. (i): $C_{n,k+1}$ counts the Γ -free matrices of size $n \times (k+1)$ without all-0 column. Each row of a Γ -free matrix

- A. starts with a 0 or
- B. starts with a 1 followed only by 0s or
- C. starts with a 1 and contains at least one more 1.

Let m denote the number of rows that starts with a 1. $m \geq 1$, since all columns contain at least one 1. We choose these m rows $\binom{n}{m}$ ways. The first $m-1$ rows has to be of type *B* since a Γ would appear. The further $(n-m+1) \times k$ elements can be filled with an arbitrary Γ -free matrix that contain in any column at least one 1.

(ii): If we argue the same way as before we obtain

$$\sum_{m=1}^n \binom{n}{m} D_{n-m+1,k},$$

but we do not count matrices that contain only type *A* and type *B* rows (and does not have type *C* rows). In this case the remainder $(n-m+1) \times k$ elements contains at least an all-0 row (the remainder of a type *B* row). Hence these matrices are not counted in the above formula.

To correct the enumeration (count the missing matrices) we must add the term

$$\sum_{m=1}^n \binom{n}{m} D_{n-m,k},$$

and obtain (ii). □

2.2. Permutation tableaux of size $n \times k$. Permutation tableaux were introduced by Postnikov [34] during his investigations of totally Grassmannian cells. They received a lot of attention after [41] Viennot showed its one-to-one correspondence to permutations, alternative tableaux and the strong connection to the PASEP model in statistical mechanics. Many bijections arised in the literature to other objects (tree-like tableaux) and to permutations in order to are use them for enumerations of permutations according to certain statistics [16], [11]. Permutation tableaux are usually defined as 01 fillings of Ferrers diagram with the next two conditions:

- (column) each column contains at least one 1.
- (1-hinge) each cell with a 1 above in the same column and to its left in the same row must contain a 1.

In the special case when the Ferrers diagram is a $n \times k$ array the definition gives actually the set $\mathcal{M}_n^k(P; c|)$, where

$$P = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Theorem 6. *We have*

$$\begin{aligned}
\text{(i)} \quad & |\mathcal{M}_n^k(P)| = B_{n,k}, \\
\text{(ii)} \quad & |\mathcal{M}_n^k(P; c)| = C_{n,k}, \\
\text{(iii)} \quad & |\mathcal{M}_n^k(P; r|c)| = D_{n,k}.
\end{aligned}$$

Proof. (i) is contained in [28] without the recognition of the relation to the poly-Bernoulli numbers. In [42] in Lemma 4.3.5 the author proves the formula also and as a corollary he receives that the number of $n \times k$ patterns of permutation diagrams is the poly-Bernoulli numbers $B_{n,k}$. For details see [42].

(ii), (iii) is proved by the obvious binomial correspondences between $|\mathcal{M}_n^k(P)|$, $|\mathcal{M}_n^k(P; c)|$, and $|\mathcal{M}_n^k(P; r|c)|$. \square

The theorem follows also from a certain bijection between permutations and permutation tableaux that we cite in a latter section.

We see that in the case of permutation tableaux the important variant is the C -relative, the one that corresponds to the restriction of the columns. This is one of the reason why we think that the introduction and investigation of the variants of poly-Bernoulli numbers is useful.

2.3. A further excluded submatrix set. Brewbaker made extensive computations considering 01 matrices with excluded patterns [9]. These suggest the following theorem that we prove by showing the recursion for the poly-Bernoulli numbers.

Theorem 7. *Let Q be the set*

$$Q = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Then we have

$$\begin{aligned}
\text{(i)} \quad & |\mathcal{M}_n^k(Q)| = B_{n,k}, \\
\text{(ii)} \quad & |\mathcal{M}_n^k(Q; c)| = C_{n,k}, \\
\text{(iii)} \quad & |\mathcal{M}_n^k(Q; r|c)| = D_{n,k}.
\end{aligned}$$

Proof. (i): Let M be a matrix in $\mathcal{M}_n^k(Q)$ and $Q_{n,k} = |\mathcal{M}_n^k(Q)|$. Let $j_1 > j_2 > \dots > j_m$ the indices of the rows of the 1 entries in the first column ($m \geq 0$). When $m = 0$ or $m = 1$ the first row does not restrict the remainder $n \times (k-1)$ entries hence it can be filled with an arbitrary matrix in $\mathcal{M}_n^{k-1}(Q)$. When $m \geq 2$ the rows j_1, j_2, \dots, j_m coincide otherwise one of the submatrix in Q would appear. It is enough to describe a $(n-m+1) \times (k-1)$ Q -free matrix in order to define M . We have:

$$Q_{n,k} = (n+1)Q_{n,k-1} + \sum_{m=2}^n \binom{n}{m} Q_{n-m+1,k-1}.$$

Hence the $Q_{n,k}$ numbers and the poly-Bernoulli numbers satisfy the same recursion. Induction proves (i).

(ii), (iii): The above proof of recursion easily extends to show the corresponding recursions for $|\mathcal{M}_n^k(Q; c)|$ and $|\mathcal{M}_n^k(Q; r|c)|$. \square

3. PERMUTATIONS

In this section we consider classes of permutations that are enumerated by the poly-Bernoulli numbers resp. their relatives. As usual let $\{1, \dots, n\} = [n]$ and S_n denote the set of permutations of $[n]$.

3.1. Vesztergombi permutations. The permutations we consider in this section are permutations that are restricted by constraints on the distance between their elements and their images. The enumeration of such permutation classes is a special case of a more general problem setting that were investigated by many authors. Given n subsets A_1, A_2, \dots, A_n of $[n]$, determine the number of permutations π such that $\pi(i) \in A_i$ for all $i \in [n]$. The problem can be formulated as enumeration of the 1-factors of a bipartite graph or as the determination of the permanent of a 01 matrix or as the number of rook-placements of a given board. In general these formulations does not make the problem easier.

We want to use the results of Lovász and Vesztergombi ([40], [32], [30]) for derivation of to (4) analogous formulas for $C_{n,k}$ and $D_{n,k}$. We recall definitions and main ideas for the sake of understanding. Detailed combinatorial proofs and analytical derivations can be found in the cited articles.

Let $f(r, n, k)$ denote the number of permutations $\pi \in S_{n+k}$ satisfying

$$-(k+r) < \pi(i) - i < n+r.$$

The main result is as follows:

Theorem 8. [40]

$$f(r, n, k) = \sum_{m=0}^n (-1)^{n+m} (m+r)! (m+r)^k \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}.$$

The original proof is analytic and depends on the solution of certain differential equation for a generating function based on the $f(r, n, k)$ numbers. The differential equations capture the recursions that follow from the expanding rules of the corresponding permanent.

Launois [29] realized the connection of this formula to the poly-Bernoulli numbers, namely that $f(2, n, k) = B_{n,k}$.

Theorem 9. [29] *Let \mathcal{V}_n^k denote the set of permutations π of $[n+k]$ such that $-k \leq \pi(i) - i \leq n$ for all $i \in [n+k]$.*

$$|\mathcal{V}_n^k| = B_{n,k}$$

Beyond the analytical derivation of the formula there are combinatorial proofs of the theorem in the literature. In [27] the authors define an explicit bijection between Vesztergombi permutations and lonesum matrices. In [30] we find a combinatorial proof for a general case that includes the theorem. For the sake of completeness we present here the direct combinatorial proof from [7].

Proof. (Theorem 9.) $|\mathcal{V}_n^k|$ is the permanent of the $(n+k) \times (n+k)$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } -k \leq i - j \leq n, \quad i = 1, \dots, n+k \\ 0 & \text{otherwise.} \end{cases}$$

The permanent of A (denoted by $\text{per} A$) counts the number of expansion terms of the matrix A which do not contain a 0 term.

The matrix A is built up of 4 blocks:

$$A = \begin{bmatrix} J_{n,k} & B_n \\ B^k & J_{k,n} \end{bmatrix},$$

where $J_{n,k} \in \{0,1\}^{n \times k}$, $J_{k,n} \in \{0,1\}^{k \times n}$ the matrices with all entries equal 1, furthermore $B_n \in \{0,1\}^{n \times n}$: $B_n(i,j) = 1$ iff $i \geq j$ and $B^k \in \{0,1\}^{k \times k}$: $B_{ij}^k = 1$ iff $i \leq j$.

For a term in the expansion of the permanent we have to select exactly one 1 from each row and each column. The number of ways of selecting 1s from the triangular matrices is given by the Stirling number of the second kind. (See proof for instance in [30].) So if a term contains m 1's from the upper left block $J_{k,n}$ ($m!$ ways), then it contains $n-m$ 1's from B_n ($\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\}$ ways); m 1's from the lower right block ($m!$ ways) and finally $k-m$ 1's from B^k ($\left\{ \begin{smallmatrix} k+1 \\ m+1 \end{smallmatrix} \right\}$ ways). The total number of terms in the expansion of $\text{per} A$ is

$$\sum_{m=1} m! \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} m! \left\{ \begin{smallmatrix} k+1 \\ m+1 \end{smallmatrix} \right\}.$$

This proves the theorem. \square

Suitable modifications of the definition of Vesztergombi permutations lead to the pB -relatives. Let \mathcal{V}_n^{k*} the set of permutations π of $[n+k]$ such that

$$-k \leq \pi(i) - i < n \quad \text{for all } i \in [n+k]$$

and \mathcal{V}_n^{k**} the set of permutations π of $[n+k]$ such that

$$-k < \pi(i) - i < n \quad \text{for all } i \in [n+k].$$

Theorem 10. ([40])

(i)

$$|\mathcal{V}_n^{k*}| = C_{n,k},$$

(ii)

$$|\mathcal{V}_n^{k**}| = D_{n,k}.$$

Proof. In these cases the blocks B_n, B^k are slightly changed. The modifications are straightforward and hence the details are omitted. \square

Corollary 1. (i)

$$C_{n,k} = \sum_{m=0}^n (-1)^{n+m} m! (m+1)^k \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\},$$

(ii)

$$D_{n,k} = \sum_{m=0}^n (-1)^{n+m} m! m^k \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\}.$$

Proof. Clearly $|\mathcal{V}_n^{k*}| = f(1, n, k-1)$ and $|\mathcal{V}_n^{k**}| = f(0, n, k)$. \square

In [32] Theorem 1 describes the asymptotic behavior of $D_{n,n}$.

Theorem 11. [32]

$$D_{n,n} \sim \sqrt{\frac{1}{2\pi(1-\ln 2)}} (n!)^2 \frac{1}{(\ln 2)^{2n}}.$$

3.2. Permutations with excedance set $[k]$. Permutations that have special restrictions on their excedance set are enumerated by the poly-Bernoulli numbers resp. their relatives. We note that the connection of this class of permutations to poly-Bernoulli numbers is not mentioned directly in the literature.

We call an index i an *excedance* (resp. *weak excedance*) of the permutation π when $\pi(i) > i$ (resp. $\pi(i) \geq i$). According that we define the set of excedances (resp. the set of weak excedances) of a permutation π as $E(\pi) := \{i | \pi(i) > i\}$ and $WE(\pi) := \{i | \pi(i) \geq i\}$. Further let define the following sets of permutations of $[n+k]$ with conditions on their excedance sets:

$$\begin{aligned}\mathcal{E}_n^k &:= \{\pi | \pi \in S_{n+k} \text{ and } WE(\pi) = [k]\}, \\ \mathcal{E}_n^{k*} &:= \{\pi | \pi \in S_{n+k} \text{ and } E(\pi) = [k]\}, \\ \mathcal{E}_n^{k**} &:= \{\pi | \pi \in S_{n+k} \text{ and } E(\pi) = [k] \text{ and } \pi(i) \neq i \text{ } \forall 1 \leq i \leq n+k\}.\end{aligned}$$

The main result in this line of research is summarized in the next theorem.

Theorem 12. *The following three statements hold:*

$$\begin{aligned}\text{(i)} \quad & |\mathcal{E}_n^k| = B_{n,k}, \\ \text{(ii)} \quad & |\mathcal{E}_n^{k*}| = C_{n,k}, \\ \text{(iii)} \quad & |\mathcal{E}_n^{k**}| = D_{n,k}.\end{aligned}$$

Proof. There are trivial bijections between these permutations and the three variants of Vesztergombi permutations. We obtain the underlying matrices of the permutation classes $\mathcal{E}_n^k, \mathcal{E}_n^{k*}, \mathcal{E}_n^{k**}$ by shifting the building blocks of the underlying matrix A of the appropriate variant of the Vesztergombi permutation. We just sketch the necessary ideas for (i). The matrix which permanent determines the size of this permutation class is built up of the following 4 blocks:

$$E = \begin{bmatrix} B_k & J_{k,n} \\ J_{n,k} & B^n \end{bmatrix},$$

where $J_{n,k} \in \{0,1\}^{n \times k}$ and $J_{k,n} \in \{0,1\}^{k \times n}$ are above (the all-1 matrices) and $B_k \in \{0,1\}^{k \times k}$: $B_k(i,j) = 1$ iff $1 \leq j \leq i \leq k$, resp. $B^n \in \{0,1\}^{n \times n}$: $B^n(i,j) = 1$ iff $1 \leq i \leq j \leq n$.

The terms in the expansion of $\text{per} E$ can be bijectively identified with the term in the corresponding expansion in the case of Vesztergombi permutations. \square

Next we connect \mathcal{E}_n^k in another way to the poly-Bernoulli family, hence we give an alternative proof of (i).

As we mentioned before permutation tableaux are well studied objects and several bijections are known between permutations and permutation tableaux. We describe a bijection between permutation tableaux and permutations that is a bijection between the sets $\mathcal{M}_n^k(P)$ and \mathcal{E}_n^k when we apply it to the subset of rectangular Ferrers shapes.

Theorem 13.

$$|\mathcal{M}_n^k(P)| = |\mathcal{E}_n^k|.$$

We modify the bijection given in [11] in order to have the following properties: the excedances of the permutation correspond to the column labels and fixed points of the permutation to the labels of empty rows. These modifications do not change the bijection essentially.

Proof. (Sketch) Consider a $n \times k$ 01 matrix that avoids the submatrices in the set P and contains a 1 in any column. We assign a permutation to this matrix the following way:

Label the positions of the rows from left to right by $[k]$, the positions of the columns from bottom to top by $[n]$. We define the zig-zag path by bouncing right or down every time we hit a 1. For i we find $\pi(i)$ by starting at the top of the column i (the left of the row i) following the zig-zag path until to boundary where we hit the row or column labeled by j and set $\pi(i) = j$.

The defined map gives a bijection between the two sets in the theorem. The details are straightforward and left to the reader. \square

In [2] the authors determined the asymptotic of $C_{n,n}$ investigated as the number of the extremal excedance set statistic.

Theorem 14. [2]

$$C_{n,n} \sim \left(\frac{1}{2 \log 2 \sqrt{(1 - \log 2)}} + o(1) \right) \left(\frac{1}{2 \log 2} \right)^{2n} (2n)!.$$

3.3. Callan permutations. Callan gave an alternative description of poly-Bernoulli numbers in a note in OEIS [33]. We repeat it, sketch the proof of his claim. We do this because Callan permutations play an important role in proving combinatorial properties of pB-relatives.

Definition 3. *Callan permutations are the permutations of $[n + k]$ in which each substring whose support belongs to $N = \{1, 2, \dots, n\}$ or $K = \{n + 1, n + 2, \dots, n + k\}$ is increasing.*

We call the elements in N *left value elements* and that of K *right value elements* and for the sake of convenience we rewrite $K \equiv \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{k}\}$ ($N = \{1, 2, \dots, n\}$). Actually we need just the distinction between the elements of the sets N and K and an order in N and K . So one can work with $N = \{0, 2, 3, \dots, n\}$ and talk about Callan permutations.

Let \mathcal{C}_n^k denote the set of Callan permutations.

Theorem 15.

$$|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} = B_{n,k}.$$

Proof. (Sketch) Let $\pi \in \mathcal{C}_n^k$. Let $\tilde{\pi} = 0\pi(\mathbf{k} + \mathbf{1})$, where 0 is a new left value and $\mathbf{k} + \mathbf{1}$ is a new right value. Divide $\tilde{\pi}$ into maximal blocks of consecutive elements such a way that each block is a subset of $\{0\} \cup N$ (left blocks) or a subset of $K \cup \{\mathbf{k} + \mathbf{1}\}$ (right blocks). The partition starts with a left block (the block of 0) and ends with a right block (the block of $\mathbf{k} + \mathbf{1}$). So the left and right block alternate, their number is the same, say $m + 1$. Describing a Callan permutation is equivalent to specifying m , a partition $\Pi_{\hat{N}}$ of $\hat{N} = \{0\} \cup N$ into $m + 1$ classes (one class is the class of 0, the other m ones are called ordinary classes), a partition $\Pi_{\hat{K}}$

of $\widehat{K} = K \dot{\cup} \{\mathbf{k} + \mathbf{1}\}$ into $m + 1$ classes (m many of them not containing $\mathbf{k} + \mathbf{1}$, the ordinary classes), and two orderings of the ordinary classes. This proves Callan's claim. \square

The role of 0 and $\mathbf{k} + \mathbf{1}$ were important. With the help of them we had the information how the left and right blocks follow each other.

Let $\mathcal{C}_n^k(*, l)$ be the set of Callan permutations of $N \dot{\cup} K$ that end with a left-value element (and hence with a left block). The star is to remind the reader that there is no assumption on the leading block of our permutation. Similarly let $\mathcal{C}_n^k(l, *)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element. Let $\mathcal{C}_n^k(l, r)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element and ends with a right element. The reader easily can define the sets $\mathcal{C}_n^k(r, *)$, $\mathcal{C}_n^k(*, r)$, $\mathcal{C}_n^k(r, l)$, $\mathcal{C}_n^k(l, l)$, and $\mathcal{C}_n^k(r, r)$.

If we take a Callan permutation, reverse the order of its blocks (leaving the order within each block) we obtain a Callan permutation too. This simple observation proves the following equalities:

$$\begin{aligned} |\mathcal{C}_n^k(*, l)| &= |\mathcal{C}_n^k(l, *)|, \\ |\mathcal{C}_n^k(r, l)| &= |\mathcal{C}_n^k(l, r)|. \end{aligned}$$

Now we state our next theorem that, gives a new interpretation of pB-relatives with the help of Callan permutations.

Theorem 16. (i)

$$C_{n,k} = |\mathcal{C}_n^k(*, l)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} \quad n \geq 1 \text{ and } k \geq 0.$$

(ii)

$$D_{n,k} = |\mathcal{C}_n^k(l, r)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} \quad n \geq 1 \text{ and } k \geq 1.$$

Proof. (i): Take a $\pi \in \mathcal{C}_n^k(*, l)$ and extend it with a starting 0 (an extra left value): $\widehat{\pi} = 0\pi$. One extra element is enough to control the structure of blocks: the block decomposition starts and ends with a left block. Let $m + 1$ the number of left blocks, m is the number of right blocks (and the number of ordinary left blocks, i.e. blocks not containing 0). The rest of the proof is a straightforward modification of the previous one.

(ii): Without any extra element we control the starting and ending block. m denotes the common number of left and right blocks. The details are left to the reader. \square

The next lemma is implicit in [6]. Since it is central for us, we present it here.

Lemma 2. *There is a bijection*

$$\varphi : \mathcal{C}_n^k(*, l) \rightarrow \mathcal{C}_{n-1}^{k+1}(*, r).$$

Proof. Take any $\pi \in \mathcal{C}_n^k(*, l)$. Find n (the largest left value) in it. It is the last element of one of the left blocks (possibly the very last element of π).

Assume that n is not the last element of π . Then it is followed by a right block R and by at least one left block. Exchange n to $\mathbf{k} + \mathbf{1}$ and move R to the end of π . The permutation that we obtain this way will be $\varphi(\pi)$.

If n is the last element of π , then exchange it to $\mathbf{k} + 1$.

In both case the described image is obviously in $\mathcal{C}_{n-1}^{k+1}(*, r)$. In order to see that φ is a bijection we need to construct its inverse. This can be done easily based on $\mathbf{k} + 1$. The details are left to the reader. \square

The lemma is obviously gives us a $\psi : \mathcal{C}_n^k(*, r) \rightarrow \mathcal{C}_{n+1}^{k-1}(*, l)$ bijection too.

In [6] this lemma was used to prove that $\sum_{k, \ell: k+\ell=n} (-1)^k B_{k, \ell} = 0$. We use the lemma for different purposes. First we combinatorially prove the symmetry of the $C_{n, k}$ numbers.

Corollary 2.

$$C_{n, k} = C_{k+1, n-1}.$$

Proof. Change the role of left and right values. The two orderings remain, hence we obtain a Callan permutation (the blocks remain the same). This leads to a bijection between $\mathcal{C}_n^k(*, l)$ and $\mathcal{C}_k^n(*, r)$. Using the previous lemma we obtain that

$$C_{n, k} = |\mathcal{C}_n^k(*, l)| = |\mathcal{C}_k^n(*, r)| = |\mathcal{C}_{k+1}^{n-1}(*, l)| = C_{k+1, n-1}.$$

\square

The next application of our lemma will be a simple connection between poly-Bernoulli numbers and its C-relative. It was proved in [26] with analytical methods. Here we present a combinatorial method.

Theorem 17. [26]

$$B_{n, k} = C_{n, k} + C_{k, n} = C_{n, k} + C_{n+1, k-1}.$$

Proof. We know that $B_{n, k} = |\mathcal{C}_n^k|$, furthermore $\mathcal{C}_n^k = \mathcal{C}_n^k(*, l) \dot{\cup} \mathcal{C}_n^k(*, r)$. We have a bijection between $\mathcal{C}_n^k(*, r)$ and $\mathcal{C}_{n+1}^{k-1}(*, l)$. Hence

$$B_{n, k} = |\mathcal{C}_n^k| = |\mathcal{C}_n^k(*, l)| + |\mathcal{C}_n^k(*, r)| = C_{n, k} + |\mathcal{C}_{n+1}^{k-1}(*, l)| = C_{n, k} + C_{n+1, k-1}.$$

\square

A similar connection is true between $C_{n, k}$ and $D_{n, k}$.

Theorem 18.

$$C_{n, k} = D_{n, k} + D_{n-1, k} + D_{n-1, k+1}.$$

Proof. We know that $C_{n, k} = |\mathcal{C}_n^k(*, l)|$, furthermore $\mathcal{C}_n^k(*, l) = \mathcal{C}_n^k(r, l) \dot{\cup} \mathcal{C}_n^k(l, l)$.

$$C_{n, k} = |\mathcal{C}_n^k(*, l)| = |\mathcal{C}_n^k(r, l)| + |\mathcal{C}_n^k(l, l)| = D_{n, k} + |\mathcal{C}_n^k(l, l)|.$$

The second term can be handled as we handled $\mathcal{C}_n^k(*, l)$ on our lemma: We present a bijection $\varphi : \mathcal{C}_n^k(l, l) \rightarrow \mathcal{C}_{n-1}^{k+1}(l, r) \dot{\cup} \mathcal{C}_{n-1}^k(r, l)$.

Let $\pi \in \mathcal{C}_n^k(l, l)$. Find the position of n (the largest left value) in π . It is the last element of one of the left blocks. If it is in the last block then simply rewrite it to $\mathbf{k} + 1$. If it is not in the last block then there is a following right block R and at least one more left block. Then also rewrite it to $\mathbf{k} + 1$ and at the same time move R to the end of π . The resulting permutation is $\varphi(\pi)$.

So far we did the same as we did in the proof of the lemma. The only problem, that the image is not necessarily in $\mathcal{C}_{n-1}^{k+1}(l, r)$. It is possible that the block of n is the first block of π and it consists of only one element. Then the lemma's idea leads to $\varphi(\pi)$ where the leading element is $\mathbf{k} + 1$. We don't want that. In this very

special case (n is the first element of π) we just erase n from π in order to obtain $\varphi(\pi)$.

Now it is clear that we defined a map with $\mathcal{C}_{n-1}^{k+1}(l, r) \dot{\cup} \mathcal{C}_{n-1}^k(r, l)$ as codomain. To see that it is a bijection we construct its inverse: If we have a permutation from $\mathcal{C}_{n-1}^k(r, l)$, then the inverse puts a starting n in front of it. If we have a permutation from $\mathcal{C}_{n-1}^{k+1}(l, r)$, then the inverse works as in our lemma.

The bijection leads to a fast end to our proof:

$$C_{n,k} = D_{n,k} + |\mathcal{C}_n^k(l, l)| = D_{n,k} + |\mathcal{C}_{n-1}^k(r, l)| + |\mathcal{C}_{n-1}^{k+1}(l, r)| = D_{n,k} + D_{n-1,k} + D_{n-1,k+1}.$$

□

4. ACYCLIC ORIENTATIONS OF BIPARTITE COMPLETE GRAPHS

The connection of poly-Bernoulli numbers to acyclic orientations of the bipartite complete graph was discovered independently in two lines of research. (*Acyclic orientation* of a graph is an assignment of direction to each edge of the graph such that there are no directed cycles.)

Cameron, Glass and Schumacher [13] investigated the problem of maximizing number of acyclic orientations of graphs with v vertices and e edges. They conjecture that if $v = 2n$ and $e = n^2$ then $K_{n,n}$ is the extremal graph. Along their research they counted the acyclic orientations of $K_{n,k}$, and established a bijection between these orientations and lonesum matrices of size $n \times k$.

In [19] the authors realized the connection of the permutations with extremal exceedance sets (see section 4) and acyclic orientations with a unique sink. Without referring to the C-relatives of poly-Bernoulli numbers they gave an interpretation of the $C_{n,k}$ numbers in terms of acyclic orientations of complete bipartite graphs. Their proof is a specialization of general statements, we reprove the version, we need, by elementary means.

We extend their results with an interpretation for $D_{n,k}$ and summarize this line of research in the next theorem. We need some notation. Let $N = \{u_1, u_2, \dots, u_n\}$, $\widehat{N} = N \cup \{u\}$, $M = \{v_1, v_2, \dots, v_k\}$, $\widehat{M} = M \cup \{v\}$ be vertex sets. Let $K_{A,B}$ denote the complete bipartite graphs on $A \dot{\cup} B$. Let \mathcal{D}_n^k denote the set of acyclic orientations of $K_{N,M}$. Let $\mathcal{D}_n^{k'}$ denote the set of acyclic orientations of $K_{N,\widehat{M}}$, where v is the only sink (vertex without outgoing edge). Let $\mathcal{D}_n^{k''}$ denote the set of acyclic orientations of $K_{\widehat{N},\widehat{M}}$, where u is the only source (vertex without ingoing edge) and v is the only sink.

Theorem 19. (i) [13]

$$|\mathcal{D}_n^k| = B_{n,k},$$

(ii) [19]

$$|\mathcal{D}_n^{k'}| = C_{n,k},$$

(iii)

$$|\mathcal{D}_n^{k''}| = D_{n,k}.$$

Proof. (i)[13]: An acyclic orientation of $K_{N,K}$ can be coded by a 01 matrix B of size $n \times k$ the following way: $b_{i,j} = 0$ whenever the edge $u_i v_j$ is oriented from u_i to v_j , and $b_{i,j} = 1$ whenever the edge $u_i v_j$ is oriented from v_j to u_i . It is easy to check that the orientation is acyclic iff the corresponding matrix B does not contain any of the submatrix of the set L , hence B lonesome. This establishes a bijection between \mathcal{D}_n^k and \mathcal{L}_n^k . The claim follows from our previous results.

(ii): Take a binary matrix coding an orientation of a complete bipartite graph $K_{A,B}$. An all-0 column (the column of vertex $w \in B$) corresponds to the information that w is a sink. Hence if we take an arbitrary orientation of $K_{N,\widehat{M}}$ from $\mathcal{D}_n^{k'}$, then its restriction to $K_{N,M}$ will be an acyclic orientation. Its coding binary matrix cannot contain an all-0 column (an all-0 column would correspond to a second sink, that cannot exist in an orientation from $\mathcal{D}_n^{k'}$). Note that there is no restriction on rows. The elements of N cannot be sinks, since the edges connecting them to v are outgoing edges.

The above argument gave us a bijection between $\mathcal{D}_n^{k'}$ and $\mathcal{L}_n^k(c)$, hence it proves our claim.

(iii): Straight forward extension of the previous proof. \square

Using classical results the connection to acyclic orientation of complete bipartite graphs immediately leads to connections to the chromatic polynomials of complete bipartite graphs. The chromatic polynomial of a graph G is a polynomial $\text{chr}_G(q)$, such that for natural number k $\text{chr}_G(k)$ gives the number of good k -colorings of G .

The famous result of Stanley [37] is that the number of the acyclic orientations of a graph is equal to the absolute value of the chromatic polynomial of the graph evaluated at -1 . Green and Zaslavsky [22] showed that the number of acyclic orientations with a given unique sink is (up to sign) the coefficient of the linear term of the chromatic polynomial (see [21] for elementary proofs). Also in [22] it is proven, that the number of acyclic orientations of a graph G with a specified uv edge, such that u is the unique source and v is the unique sink is the derivative of the chromatic polynomial evaluated at 1 (the necessary signing is taken), the so called ‘‘Crapo’s beta invariant’’. Again [21] present an elementary discussion of this result.

By putting together the information quoted above we obtain the following theorem.

Theorem 20. (i) [13]

$$B_{n,k} = (-1)^{n+k} \text{chr}_{K_{n,k}}(-1),$$

(ii) [19]

$$C_{n,k} = (-1)^{n+k} [q] \text{chr}_{K_{n,k+1}}(q),$$

(iii)

$$D_{n,k} = (-1)^{n+k} \left(\frac{d}{dq} \text{chr}_{K_{n+1,k+1}} \right) (1).$$

The chromatic polynomials of complete bipartite graphs are well understood. We list a few results on this topic.

The exponential generating function of the chromatic polynomial of $K_{n,k}$ [38] Ex. 5.6:

$$\sum_{n \geq 0} \sum_{k \geq 0} \text{chr}_{K_{n,k}}(q) \cdot \frac{x^n}{n!} \frac{y^k}{k!} = (e^x + e^y - 1)^q$$

Several formulas for the chromatic polynomial of complete bipartite graphs are known (for example [39], [19], [23]):

$$\text{chr}_{K_{n,k}}(q) = \sum_{i=0}^n \sum_{j=0}^k \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} (q)_{i+j},$$

where $(q)_\ell = q(q-1)(q-2)\dots(q-\ell+1)$, the “falling factorial”.

$$\text{chr}_{K_{n,k}}(q) = \sum_{m \geq 0} \left(\sum_{i=0}^n \sum_{j=0}^k s(i+j, m) \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} \right) q^m,$$

where $s(n, k)$ is the (signed) Stirling number of the first kind.

Simple arithmetic leads to the following theorem:

Theorem 21. (i)

$$B_{n,k} = (-1)^{n+k} \sum_{m=0}^{n+k} \sum_{i=0}^n \sum_{j=0}^k (-1)^m s(i+j, m) \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix},$$

(ii)

$$C_{n,k} = (-1)^{n+k} \sum_{i=0}^n \sum_{j=0}^k s(i+j, 1) \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{Bmatrix} k+1 \\ j \end{Bmatrix},$$

(iii)

$$D_{n,k} = (-1)^{n+k} \sum_{l=0}^{n+k+2} \sum_{i=0}^n \sum_{j=0}^k l s(i+j, l) \begin{Bmatrix} n+1 \\ i \end{Bmatrix} \begin{Bmatrix} k+1 \\ j \end{Bmatrix}.$$

We mention that the formula in (ii) is implicit in [19], without mentioning the poly-Bernoulli connection.

5. ALGORITHMS FOR GENERATING THE SERIES

In this section we recall algorithms that computes the arrays $B_{n,k}$, $C_{n,k}$ and $D_{n,k}$ by similar simple rules as Pascal’s triangle the binomial coefficients. We will see that in this context the relatives $D_{n,k}$ arise naturally.

This line of research was initiated by the Akiyama–Tanigawa algorithms, that generates the Bernoulli numbers. Let define the array $a_{n,i}$ recursively (based on $\{a_{0,i}\}$) by the rule:

$$a_{n+1,i} = (i+1)(a_{n,i} - a_{n,i+1}).$$

Akiyama–Tanigawa proved that if the initial sequence is $a_{0,i} = \frac{1}{i}$ then $a_{n,0}$ are the n -th Bernoulli numbers. Let denote by AT the transformation $\{a_{0,i}\} \rightarrow \{a_{n,0}\}$. Akiyama–Tanigawa’s theorem says that $AT(\{1/(i+1)\}_i) = \{B_i\}_i$ (with $B_1 = \frac{1}{2}$).

Kaneko showed [25] that for any initial sequence $a_{0,i}$ it holds

$$a_{n,0} = \sum_{i=0}^n (-1)^i i! \begin{Bmatrix} n+1 \\ i+1 \end{Bmatrix} a_{0,i}.$$

Based on the sieve formulas and simple arithmetic we obtain the following theorem.

Theorem 22. (i) [25]

$$AT(\{(i+1)^k\}_i) = \{(-1)^i C_{i,k}\}_i,$$

(ii)

$$AT(\{i^k\}_i) = \{(-1)^i D_{i,k}\}_i$$

The poly-Bernoulli numbers itself can be generated also by such simple rules, though the recursive rule has to be modified for that. Chen [14] presents the variant of the algorithm with these changes:

$$b_{n+1,i} = ib_{n,i} - (i+1)b_{n,i+1},$$

Chen shows (Proposition 2.) that

$$b_{n,0} = \sum_{i=0}^n (-1)^n i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} b_{0,i}.$$

Let denote BT transformation $\{b_{0,i}\} \rightarrow \{b_{n,0}\}$ the transformation based on the modified recursive rule. One consequence of Chen's theorem ([14] Theorem 1.) is $BT(\{1/(i+1)\}_i) = \{B_i\}_i$ (with $B_1 = -\frac{1}{2}$) and another is that

Theorem 23.

$$BT(\{(i+1)^k\}_i) = \{(-1)^i B_{i,k}\}_i$$

6. DIAGONAL SUM OF POLY-BERNOULLI NUMBERS

The diagonal sum of poly-Bernoulli numbers resp. their relatives arise in analytical, number theoretical and combinatorial investigations [33], [26], [2]. However a nice formula is still missing. The diagonal sum of the poly-Bernoulli numbers

$$\sum_{n+k=N} B_{n,k}$$

are referred in OEIS [33] A098830:

$$1, 2, 4, 10, 32, 126, 588, 3170, \dots$$

The diagonal sum of the C -relatives are also referred in OEIS [33] A136127:

$$1, 2, 5, 16, 63, 294, 1585 \dots$$

The simple arithmetic relation between $B_{n,k}$ and $C_{n,k}$ of theorem 17. implies that (except the first entry) the A098830 is exactly the double of A136127.

From the combinatorial point of view the diagonal sum enumerates of course sets of the combinatorial objects we listed in this paper before. However there are combinatorial objects where this sum itself appears naturally: there is no reason for the division of the basic set of size N into two sets of size n and k with $n+k=N$. Here we mention some of them.

The ascending-to-max property [20] is one of the characteristic property of permutations that are suffix arrays of binary words. Suffix arrays play an important role in efficient searching algorithms of given patterns in a text.

Cycles without stretching pairs [2] received attention because of their connection to a result of Sharkovsky in discrete dynamical systems. The occurrence of a stretching pair within a periodic orbit implies turbulence [17]. In [17] we find also the description of strong connections to permutations that avoid $(21-34)$ or $(34-21)$ as generalized patterns.

The introduction of the combinatorial non-ambiguous trees [5] that are compact embeddings of binary trees in a grid, was motivated by enumeration of parallelogram polynomials. Non-ambiguous trees are actually special cases of tree-like tableaux, objects that are in one-to-one correspondence with permutation tableaux.

From the analytical results we recall here an interesting connection to the central binomial sum:

$$CB(k) = \sum_{n \geq 1} \frac{n^k}{\binom{2n}{n}}$$

Borwein and Girgensohn ([10] section 2.) showed that

$$CB(N) = P_N + Q_N \frac{\pi}{\sqrt{3}},$$

where P_N and Q_N are explicitly given rationals.

Stephan's computations suggest the following interesting conjecture [33]:

Conjecture 1.

$$\sum_{n+k=N} B_{n,k} = 3P_N.$$

Based on the explicit formula that was given in [10] we can reformulate the Stephan's conjecture:

$$\sum_{n+k=N} B_{n,k} = 3P_N = (-1)^{N+1} \frac{1}{2} \sum_{j=1}^{N+1} (-1)^j j! \left\{ \begin{matrix} N+1 \\ j \end{matrix} \right\} \frac{\binom{2j}{j}}{3^{j-1}} \sum_{i=0}^{j-1} \frac{3^i}{(2i+1)\binom{2i}{i}}.$$

It would be interesting to prove the conjecture or/and to find a simple expression for the diagonal sum.

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